Sequential Investment in Pollution Control Equipment under Uncertainty

Motoh Tsujimura, Doshisha University, mtsujimu@mail.doshisha.ac.jp

1 Overview

This paper investigates the investment in pollution control capital under uncertainty. We assume that a firm's output generates a pollution by-product and the pollution reduces the productivity of capital. The dynamics of pollution is assumed to be governed by stochastic differential equation. Then, the firm has to invest in a pollution control capital in order to reduce the pollutant and it incurs cost. The firm also pays environmental tax which is set at rate per unit of emissions. In this paper, we assume that the firm can invest it as needed. Then, the firm's problem is to choose the investment timing under uncertainty and is formulated a singular stochastic control problem. We solve the firm's problem by using variational inequalities and obtain the optimal investment strategy which is described by the threshold to invest in the pollution control equipment. Furthermore, we conduct a comparative static analysis on some parameters.

2 The Model

Suppose that a firm produces a single output by using a variable input L like labor and sells it in competitive market. The input price w and the output price p are assumed to be constants. The firm's production function F is given by $F(L_t) = aL_t^\gamma$, where a is a constant that reflects the level of the production technology and γ is the output elasticity of the variable input. The production process generates pollution emissions E proportional to output level. The pollution emissions E is given by $E_t = \eta(K_t)F(L_t)$, where η is the emission coefficient function of the stock of abatement capacity K. In this paper, we specify η as $\eta(K_t) = bK_t^{-\lambda}$, where b is a constant that reflects the level of the abatement technology and λ is the parameter. The firm has to pay tax τ per unit of emissions. The firm then invests in the abatement capacity to reduce the pollution emissions. Let I_t be the cumulative purchase of abatement equipment until time t. The firm can buy the abatement equipment at any time t at constant price c. As in Pham (2006), the firm's abatement capacity evolves according to:

$$\mathrm{d}K_t = -\delta K_t \mathrm{d}t + \sigma K_t \mathrm{d}W_t + \mathrm{d}I_t, \quad K_{0^-} = k \ (>0), \tag{1}$$

where δ is a constant depreciation rate of the abatement equipment and σ is its volatility. W_t is a standard Brownian motion.

We assume that the variable input is assumed to be costlessly and instantaneously adjusted. Then, the firm's maximized instantaneous operating profit π is derived as $\pi(K_t) = (p - \tau b K_t^{-\lambda})^\alpha h$, where $\alpha := 1/(1-\gamma)$ and $h := \alpha^\alpha(\alpha-1)^{-\alpha-1}w^{1-\alpha}a^{-\alpha}$. Then the firm's expected discounted profit $J(k;I)$ is given by:

$$J(k;I) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(K_t)\mathrm{d}t - c\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}I_t\right], \tag{2}$$

where r is a discount rate, $I = \{I_t\}_{t\geq 0}$ is the investment strategy. Therefore, the firm's problem is to maximize the expected discounted profit:

$$V(k) = \sup_I J(k;I) = J(k;I^*). \tag{3}$$

3 Variational Inequalities

From the formulation of the firm's problem (3), it is able to guess that, under an optimal investment strategy, the firm invests in the pollutant abatement capacity whenever the stock of abatement capacity K reaches a threshold $\underline{k}$. In order to verify this conjecture, we solve the firm's problem (3) by using variational inequalities.

The following relations are called the variational inequalities for the agent problem (3):

$$\mathcal{L}V(k) + \pi(k) \leq 0, \tag{4}$$

$$V'(k) \leq c, \tag{5}$$

$$[\mathcal{L}V(k) + \pi(k)][V'(k) - c] = 0, \tag{6}$$

where $\mathcal{L}$ is the operator defined by $\mathcal{L} := \frac{1}{2}\sigma^2 k^2 \frac{\mathrm{d}^2}{\mathrm{d}k^2} - \delta k \frac{\mathrm{d}}{\mathrm{d}k} - r$.

Theorem 3.1. *A solution of the variational inequalities is equivalent to the value function of the firm's problem and the policy induced by variational inequalities is optimal.*

For analytical simplicity, we assume that $\gamma = 1/2$ and $\lambda = 1$. Then I have $\alpha = 2$. For $k > \underline{k}$, the variational inequalities (4)–(6) lead to the following ordinary differential equation: $\frac{1}{2}\sigma^2 k^2 \phi''(k) - \delta k \phi'(k) - r\phi(k) + \pi(k) = 0$. Then, the general solution is:

$$\phi(k) = A_2 k^{\beta_2} + \frac{p^2 h}{r} - \frac{2p\tau b k^{-1} h}{\rho_1} + \frac{\tau^2 b^2 k^{-2} h}{\rho_2}, \qquad k > \underline{k}. \tag{7}$$

β_2 is the negative solution to the characteristic equation: $\Gamma(\beta) := \frac{1}{2}\sigma^2\beta(\beta - 1) - \delta\beta - r = 0$. Two unknowns A_2 and $\underline{k}$ are determined by the following simultaneous equations:

$$\phi'(\underline{k}) = c, \tag{8}$$

$$\phi''(\underline{k}) = 0. \tag{9}$$

The results of comparative static analysis will be presented in the conference.

4 Conclusion

This paper investigates the firm's pollutant abatement investment problem. To solve it, we formulate it as the singular stochastic control problem and use the variational inequalities to solve the problem. We then show the optimal investment strategy which is characterized the threshold $\underline{k}$. That is, the firm invests in the abatement capacity whenever the stock of abatement capacity reaches the threshold. Furthermore, we will show some comparative static analysis.

References

Pham, H. (2006), Explicit Solution to an Irreversible Investment Model with a Stochastic Production Capacity, in: Kabanov, Y, R. Lipster and J. Stoyanov (eds.), *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift*, 547–566, Springer-Verlag, Berlin.